IX. *On the Calculation of Attractions, and the Figure of the Earth.*

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THE principal object of the calculations contained in the following paper, is to investigate the figure which a fluid, consisting of portions varying in density according to any given law, would assume, when every particle is acted upon by the attraction of every other and by a centrifugal force arising from rotatory motion. To what extent this may have been the original condition of the earth, is a doubtful question; and although observation does not fully warrant this supposition of the regular arrangement of parts, it has necessarily been made the foundation of most of the mathematical calculations connected with the investigation. Before proceeding to this problem, it is necessary to calculate the attraction of a body of any given figure, and consisting of strata, varying in their densities according to any given law; and it is in this problem that the principal difficulty lies. The elegant method of solution discovered by LAPLACE is well known; and I have followed his steps as far as the point where the equation, known by his name, first appears. In order to illustrate the nature of the deviation which I have there made, it will be necessary to mention some of the principal steps of the two methods.

By means of a theorem, which LAPLACE laid down as true of all spheroids that differ but little from spheres, and the properties of the integral of the equation referred to, he was enabled to substitute the easy rules of differentiation for the more complicated inverse processes, and thus to compute the attraction of that class of figures. It has, however, been since discovered by Mr. IVORY, that this theorem is true only of spheroids of a particular kind; and, consequently, to this kind the solution of the problem is restricted. This defect, and the indirectness of his analysis, led other mathematicians to consider the question; and, in 1811, Mr. IVORY published his method, which has the great advantage of being more direct, though equally limited.

The method given in the following paper does not appear to be confined in its operation to any particular class of spheroids; since the coefficients of the series, into which the required function is developed, are determined absolutely, without any reference to the form of the spheroid to which they are about to be applied. The principal change consists in the different manner of treating this partial differential equation. LAPLACE and the subsequent writers on this equation, both as applied to the

calculation of attractions and the mathematical theory of electricity, suppose the coefficients of every term of the series to be expanded into another series of the sines and cosines of multiple arcs; and they avail themselves of the property which these terms possess of vanishing, in certain cases, when integrated between certain limits. The success of this plan, however, depends upon the restricting hypothesis above referred to, that the radius vector of the surface of the body is capable of expansion in a series of terms, each of which satisfies LAPLACE's equation. The following method shows that the coefficient of the general term of the first series is independent of one of the variables, and thus dispenses with the second series of expansions. This result I have arrived at, by first obtaining the integral of LAPLACE's equation in its most general form, and deducing the arbitrary functions introduced therein, from considerations which enter previous to the equation of the surface of the attracting body. These coefficients being known, it is evident that the attraction of any homogeneous body on a point within or without it may be immediately found when the equation of its surface is given, since it then depends only on a series of explicit and definite integrations of known functions, which can always be effected, at least approximately. From this, the attraction of a heterogeneous body, similarly circumstanced, may be found by the usual method of dividing it into concentric layers, and summing the several attractions of these, deduced as above.

By substituting the attraction so obtained, in the equation of equilibrium of a fluid body, CLAIRAUT's theorem is immediately deduced; and, from a peculiarity in the functions representing the attraction, it will be seen, that the same principles with longer processes may be carried on indefinitely, without the necessity of actually determining the precise form of those functions.

The restricted species of spheroid above referred to, comprises all surfaces of revolution; so that it is sufficiently extensive for most practical purposes; but the integration of LAPLACE's equation renders the analysis more direct, and the theory more complete.

On the General Problem of Attractions.

1. Let ϱ represent the density of a body at the point (x, y, z); and let f, g, h be the coordinates of a particle attracted by the body, parallel respectively to the axes x, y, z; then, if the power of attraction be inversely as the square of the distance, the resolved part of the attraction of the body, parallel to

$$x \text{ is} \iiint \frac{\varrho (f-x)\, dx\, dy\, dz}{\{(f-x)^2 + (g-y)^2 + (h-z)^2\}^{\frac{3}{2}}},$$

$$y \text{ is} \iiint \frac{\varrho (g-y)\, dx\, dy\, dz}{\{(f-x)^2 + (g-y)^2 + (h-z)^2\}^{\frac{3}{2}}},$$

$$z \text{ is} \iiint \frac{\varrho (h-z)\, dx\, dy\, dz}{\{(f-x)^2 + (g-y)^2 + (h-z)^2\}^{\frac{3}{2}}},$$

the limits of integration being determined by the equation to the surface of the body.

2. Let V represent the sum of the products of each particle by the reciprocal of its distance from the attracted point.

Then $V = \iiint \frac{\varrho\, dx\, dy\, dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{1}{2}}}$; and, by differentiating V, we obtain the well-known property $\frac{d^2 V}{df^2} + \frac{d^2 V}{dg^2} + \frac{d^2 V}{dh^2} = 0$, or $-4\pi\varrho'$, according as the attracted particle is not or is within the attracting mass; ϱ' being in the latter case the density of the attracted particle*. By transforming these equations to polar coordinates, we obtain

$$\frac{d^2 V}{dr^2} + \frac{2}{r}\frac{dV}{dr} + \frac{1}{r^2}\frac{d^2 V}{d\theta^2} + r^2 \cot\theta \frac{dV}{d\theta} + \frac{1}{r^2 \sin^2\theta}\frac{d^2 V}{d\varphi^2} = 0, \text{ or } -4\pi\varrho',$$

and

$$V = \int_0^r \int_0^\pi \int_0^{2\pi} \frac{\varrho\, r'^2\, dr' \sin\theta'\, d\theta'\, d\varphi'}{\{r^2 + r'^2 - 2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi'))\}^{\frac{1}{2}}};$$

where $r^2 = f^2 + g^2 + h^2$, $\cos\theta = \frac{h}{\sqrt{(f^2+g^2+h^2)}}$, $\tan\varphi = \frac{g}{f}$; and similar expressions are true of r' θ' and φ' in terms of x, y, z.

Put $\cos\theta = \mu$, and $\cos\theta' = \mu'$, and they become

$$r\frac{d^2(rV)}{dr^2} + \frac{d}{d\mu}\left((1-\mu^2)\frac{dV}{d\mu}\right) + \frac{1}{1-\mu^2}\frac{d^2 V}{d\varphi^2} = 0, \text{ or } -4\pi\varrho' r^2 \dagger. \quad . \quad . \quad . \quad . \quad (1.)$$

$$V = \int_0^r \int_{-1}^1 \int_0^{2\pi} \frac{\varrho\, r'^2\, dr'\, d\mu'\, d\varphi'}{\{r^2 + r'^2 - 2rr'(\mu\mu' + \sqrt{(1-\mu^2)}\sqrt{(1-\mu'^2)}\cos(\varphi-\varphi'))\}^{\frac{1}{2}}} \quad . \quad (2.)$$

3. Expansion by the binomial theorem shows that

$$\{r^2 + r'^2 - 2rr'(\mu\mu' + \sqrt{(1-\mu^2)}\sqrt{(1-\mu'^2)}\cos(\varphi-\varphi'))\}^{\frac{1}{2}}$$

may be expressed either in powers of r or of r'; thus

$$P_0\frac{1}{r'} + P_1\frac{r}{r'^2} + \ldots + P_n\frac{r^n}{r'^{n+1}} + \ldots, \text{ or } P_0\frac{1}{r} + P_1\frac{r'}{r^2} + P_2\frac{r'^2}{r^3} + \ldots + P_n\frac{r'^n}{r^{n+1}} + \ldots,$$

where P_n is a symmetrical function of μ, $\sqrt{(1-\mu^2)}\cos\varphi$, $\sqrt{(1-\mu^2)}\sin\varphi$ on the one part, and μ', $\sqrt{(1-\mu'^2)}\cos\varphi'$, $\sqrt{(1-\mu'^2)}\sin\varphi'$ on the other.

By substituting the first expansion in (2.), and the value of V so obtained in (1.), we have a series of equations

$$\int_0^r \int_{-1}^1 \int_0^{2\pi} \varrho \left\{\frac{d}{d\mu}\left((1-\mu^2)\frac{dP_n}{d\mu}\right) + \frac{1}{1-\mu^2}\frac{d^2 P_n}{d\varphi^2} + n(n+1)P_n\right\} \frac{dr'\, d\mu'\, d\varphi'}{r'^{n-1}}$$
$$= 0, \text{ or } -4\pi\varrho',$$

except when $n = 2$; and in all cases

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP_n}{d\mu}\right) + \frac{1}{1-\mu^2}\frac{d^2 P_n}{d\varphi^2} + n(n+1)P_n = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

which is the equation of LAPLACE's coefficients.

* Vide Pratt. Mec. Phil., § 168. LAPLACE, Méc. Cél. liv. iii.

† Vide Pratt. Mec. Phil., § 169.

4. This equation was not integrated; but by a skilful use of its properties, the problem of attractions was greatly simplified by LAPLACE. He laid down a theorem, respecting the surfaces of all spheroids of small deviation, that their radii vectores might be developed into series, every term of which would satisfy the above equation; and he also gave a method of expansion. By means of this theorem, the problem could be solved for spheroidal bodies which differ but little from spheres; but its generality has been greatly restricted by the researches of subsequent writers*, by whom it has been shown that it is true only for bodies whose radii are expressible in *rational and integral functions of* μ', $\sqrt{(1-\mu'^2)}\cos\varphi'$, $\sqrt{(1-\mu'^2)}\sin\varphi'$. Among these are the ellipsoid and elliptical spheroid, and a large class of other spheroids. In these papers I have adopted a different proceeding; I integrate the equation itself generally, and determine the arbitrary functions contained in the integrals by the circumstances of the problem itself. In consequence of the peculiar form which P_n then takes, V may be found by effecting the operations indicated, which are only explicit integrations.

5. I shall now proceed to integrate this equation.

Consider μ and φ as functions of two new variables X and Y, to be determined from the equations,

$$dX = \frac{dX}{d\varphi}(d\varphi + k\,d\mu)$$

$$dY = \frac{dY}{d\varphi}(d\varphi + k'\,d\mu),$$

where k and k' are the roots of the equation $(1-\mu^2)k^2 + \frac{1}{1-\mu^2} = 0$. These roots are $\frac{\pm\sqrt{-1}}{1-\mu^2}$, whence we obtain

$$X = \varphi + \tfrac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu}, \text{ and } Y = \varphi - \tfrac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu}. \quad . \quad . \quad (4.)$$

$$\frac{d^2P_n}{d\mu^2} = \left(\frac{dX}{d\mu}\right)^2\frac{d^2P_n}{dX^2} + 2\frac{dX}{d\mu}\frac{dY}{d\mu}\frac{d^2P_n}{dX\,dY} + \left(\frac{dY}{d\mu}\right)^2\frac{d^2P_n}{dY^2} + \frac{d^2X}{d\mu^2}\frac{dP_n}{dX} + \frac{d^2Y}{d\mu^2}\frac{dP_n}{dY},$$

$$= -\frac{1}{(1-\mu^2)^2}\frac{d^2P_n}{dX^2} + \frac{2}{(1-\mu^2)^2}\frac{d^2P_n}{dX\,dY} - \frac{1}{(1-\mu^2)^2}\frac{d^2P_n}{dY^2} + \frac{2\mu\sqrt{-1}}{(1-\mu^2)^2}\left(\frac{dP_n}{dX} - \frac{dP_n}{dY}\right);$$

$$\frac{dP_n}{d\mu} = \frac{dX}{d\mu}\frac{dP_n}{dX} + \frac{dY}{d\mu}\frac{dP_n}{dY} = \frac{\sqrt{-1}}{1-\mu^2}\left(\frac{dP_n}{dX} - \frac{dP_n}{dY}\right);$$

$$\frac{d^2P_n}{d\varphi^2} = \left(\frac{dX}{d\varphi}\right)^2\frac{d^2P_n}{dX^2} + 2\frac{dX}{d\varphi}\frac{dY}{d\varphi}\frac{d^2P_n}{dX\,dY} + \left(\frac{dY}{d\varphi}\right)^2\frac{d^2P_n}{dY^2} + \frac{d^2X}{d\varphi^2}\frac{dP_n}{dX} + \frac{d^2Y}{d\varphi^2}\frac{dP_n}{dY},$$

$$= \frac{d^2P_n}{dX^2} + 2\frac{d^2P_n}{dX\,dY} + \frac{d^2P_n}{dY^2}.$$

Substituting these in (3.), we obtain

$$\frac{4}{1-\mu^2}\frac{d^2P_n}{dX\,dY} + n(n+1)P_n = 0.$$

* See two articles by Mr. IVORY in the Philosophical Transactions, 1812.

From (4.), by subtraction,

$$\mathrm{X}-\mathrm{Y}=\sqrt{-1}\log\frac{1+\mu}{1-\mu};$$

whence

$$\mu=\frac{\varepsilon^{-(\mathrm{X}-\mathrm{Y})\sqrt{-1}}-1}{\varepsilon^{-(\mathrm{X}-\mathrm{Y})\sqrt{-1}}+1},$$

and

$$1-\mu^2=\frac{4\,\varepsilon^{-(\mathrm{X}-\mathrm{Y})\sqrt{-1}}}{\left(\varepsilon^{-(\mathrm{X}-\mathrm{Y})\sqrt{-1}}+1\right)^2}=\frac{4}{\left(\varepsilon^{-\frac{1}{2}(\mathrm{X}-\mathrm{Y})\sqrt{-1}}+\varepsilon^{\frac{1}{2}(\mathrm{X}-\mathrm{Y})\sqrt{-1}}\right)^2}=\frac{1}{\cos^2\frac{\mathrm{X}-\mathrm{Y}}{2}};$$

consequently

$$\frac{d^2\mathrm{P}_n}{d\mathrm{X}\,d\mathrm{Y}}+\frac{n\,(n+1)\,\mathrm{P}_n}{4\cos^2\frac{\mathrm{X}-\mathrm{Y}}{2}}=0.\quad \text{Let } n\,(n+1)=a.$$

Let

$$\frac{d\mathrm{P}_n}{d\mathrm{Y}}=v,\ \text{then}\ \frac{dv}{d\mathrm{X}}+\frac{a\,\mathrm{P}_n}{4\cos^2\frac{\mathrm{X}-\mathrm{Y}}{2}}=0,\ \text{and}\ \mathrm{P}_n=-\frac{dv}{d\mathrm{X}}\,\frac{4}{a}\cos^2\frac{\mathrm{X}-\mathrm{Y}}{2};$$

$$v=\frac{d\mathrm{P}_n}{d\mathrm{Y}}=-\frac{d^2v}{d\mathrm{X}\,d\mathrm{Y}}\,\frac{4}{a}\cos^2\frac{\mathrm{X}-\mathrm{Y}}{2}-\frac{dv}{d\mathrm{X}}\,\frac{4}{a}\cos\frac{\mathrm{X}-\mathrm{Y}}{2}\sin\frac{\mathrm{X}-\mathrm{Y}}{2},$$

or

$$\frac{d^2v}{d\mathrm{X}\,d\mathrm{Y}}+\frac{dv}{d\mathrm{X}}\tan\frac{\mathrm{X}-\mathrm{Y}}{2}+\frac{a\,v}{4\cos^2\frac{\mathrm{X}-\mathrm{Y}}{2}}=0.$$

Let

$$\frac{dv}{d\mathrm{Y}}+v\tan\frac{\mathrm{X}-\mathrm{Y}}{2}=t.$$

Then

$$\frac{d^2v}{d\mathrm{X}\,d\mathrm{Y}}+\frac{dv}{d\mathrm{X}}\tan\frac{\mathrm{X}-\mathrm{Y}}{2}+\frac{v}{2\cos^2\frac{\mathrm{X}-\mathrm{Y}}{2}}=\frac{dt}{d\mathrm{X}},$$

and

$$\frac{dt}{d\mathrm{X}}+\frac{a-2}{4}\,\frac{v}{\cos^2\frac{\mathrm{X}-\mathrm{Y}}{2}}=0;\ \text{or}\ v=-\frac{4}{a-2}\cos^2\frac{\mathrm{X}-\mathrm{Y}}{2}\,\frac{dt}{d\mathrm{X}};$$

whence

$$\frac{dv}{d\mathrm{Y}}=-\frac{4}{a-2}\cos^2\frac{\mathrm{X}-\mathrm{Y}}{2}\,\frac{d^2t}{d\mathrm{X}\,d\mathrm{Y}}-\frac{4}{a-2}\cos\frac{\mathrm{X}-\mathrm{Y}}{2}\sin\frac{\mathrm{X}-\mathrm{Y}}{2}\,\frac{dt}{d\mathrm{X}};$$

$$t=-\frac{4}{a-2}\cos^2\frac{\mathrm{X}-\mathrm{Y}}{2}\,\frac{d^2t}{d\mathrm{X}\,d\mathrm{Y}}-\frac{8}{a-2}\cos\frac{\mathrm{X}-\mathrm{Y}}{2}\sin\frac{\mathrm{X}-\mathrm{Y}}{2}\,\frac{dt}{d\mathrm{X}},$$

or

$$\frac{d^2t}{d\mathrm{X}\,d\mathrm{Y}}+\frac{dt}{d\mathrm{X}}\,2\tan\frac{\mathrm{X}-\mathrm{Y}}{2}+t\,\frac{a-2}{4\cos^2\frac{\mathrm{X}-\mathrm{Y}}{2}}=0.$$

Let

$$\frac{dt}{d\mathrm{Y}}+2\,t\tan\frac{\mathrm{X}-\mathrm{Y}}{2}=q,$$

and by repeating a similar process, we obtain

$$\frac{d^2 q}{d\mathrm{X}\, d\mathrm{Y}} + \frac{dq}{d\mathrm{X}} 3 \tan \frac{\mathrm{X}-\mathrm{Y}}{2} + q \frac{a-6}{4\cos^2 \frac{\mathrm{X}-\mathrm{Y}}{2}} = 0.$$

By observing the assumptions here made, and the results obtained, we find that in the first assumption $\left(\frac{d\mathrm{P}_n}{d\mathrm{Y}} = v\right)$, the coefficient of $\mathrm{P}_n \tan \frac{\mathrm{X}-\mathrm{Y}}{2}$ is 0; in the second, that of $v \tan \frac{\mathrm{X}-\mathrm{Y}}{2} = 1$; and so on, in the order of the natural numbers; and in the results, the numerical coefficients are $1, \frac{a}{4}$; $2, \frac{a-2}{4}$; $3, \frac{a-6}{4}$ ... generally $n, \frac{a-n(n-1)}{4}$. I shall prove this in the general case, by showing that if it is true of one value of n (as we see it is), it is true of the next value, and so on. Let the $(n-1)$th substitution give

$$\frac{d^2 \varrho}{d\mathrm{X}\, d\mathrm{Y}} + \frac{d\varrho}{d\mathrm{X}} (n-1) \tan \frac{\mathrm{X}-\mathrm{Y}}{2} + \varrho \frac{a-(n-1)(n-2)}{4\cos^2 \frac{\mathrm{X}-\mathrm{Y}}{2}} = 0,$$

and let

$$\frac{d\varrho}{d\mathrm{Y}} + (n-1)\, \varrho \tan \frac{\mathrm{X}-\mathrm{Y}}{2} = s:$$

then, as before,

$$\frac{d^2 \varrho}{d\mathrm{X}\, d\mathrm{Y}} + (n-1) \tan \frac{\mathrm{X}-\mathrm{Y}}{2} \cdot \frac{d\varrho}{d\mathrm{X}} + \frac{n-1}{2} \frac{1}{\cos^2 \frac{\mathrm{X}-\mathrm{Y}}{2}} \cdot \varrho = \frac{ds}{d\mathrm{X}};$$

and, therefore,

$$\frac{ds}{d\mathrm{X}} + \varrho \frac{a-(n-1)n}{4\cos^2 \frac{\mathrm{X}-\mathrm{Y}}{2}} = 0, \text{ and } \varrho = -\frac{ds}{d\mathrm{X}} \frac{4}{a-(n-1)n} \cos^2 \frac{\mathrm{X}-\mathrm{Y}}{2};$$

$$\frac{d\varrho}{d\mathrm{Y}} = \frac{d^2 s}{d\mathrm{X}\, d\mathrm{Y}} \frac{4}{a-(n-1)n} \cos^2 \frac{\mathrm{X}-\mathrm{Y}}{2} - \frac{ds}{d\mathrm{X}} \frac{4\cos \frac{\mathrm{X}-\mathrm{Y}}{2} \cdot \sin \frac{\mathrm{X}-\mathrm{Y}}{2}}{a-(n-1)n}.$$

Consequently

$$-\frac{d^2 s}{d\mathrm{X}\, d\mathrm{Y}} \frac{4}{a-(n-1)n} \cos^2 \frac{\mathrm{X}-\mathrm{Y}}{2} - \frac{ds}{d\mathrm{X}} \frac{4}{a-(n-1)n} \cos \frac{\mathrm{X}-\mathrm{Y}}{2} \cdot \sin \frac{\mathrm{X}-\mathrm{Y}}{2}$$
$$- \frac{ds}{d\mathrm{X}} \frac{4(n-1)}{a-(n-1)n} \cos \frac{\mathrm{X}-\mathrm{Y}}{2} \cdot \sin \frac{\mathrm{X}-\mathrm{Y}}{2} = s,$$

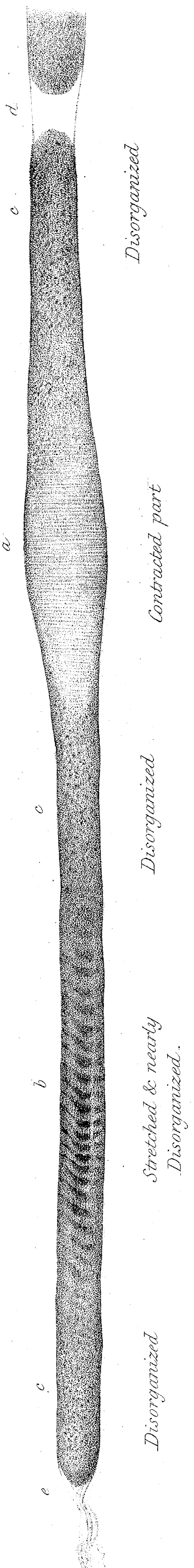
or

$$\frac{d^2 s}{d\mathrm{X}\, d\mathrm{Y}} \frac{4}{a-(n-1)n} \cos^2 \frac{\mathrm{X}-\mathrm{Y}}{2} + \frac{ds}{d\mathrm{X}} \frac{4n}{a-(n-1)n} \cos \frac{\mathrm{X}-\mathrm{Y}}{2} \cdot \sin \frac{\mathrm{X}-\mathrm{Y}}{2} + s = 0;$$

that is,

$$\frac{d^2 s}{d\mathrm{X}\, d\mathrm{Y}} + \frac{ds}{d\mathrm{X}} n \tan \frac{\mathrm{X}-\mathrm{Y}}{2} + s \frac{a-n(n-1)}{4} \cos^2 \frac{\mathrm{X}-\mathrm{Y}}{2} = 0,$$

and, therefore, the law of coefficients, as above stated, is correct.



Tetanus — part of P. Fasc. of Complexus (ecchymosed) Feb. 19. 1841.

Magnified about 300 diameters.

W. Bowman ad. nat. del.

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Restoring the value of a, we get

$$\frac{d^2 s}{d\mathrm{X}\, d\mathrm{Y}} + \frac{ds}{d\mathrm{X}} n \tan\frac{\mathrm{X}-\mathrm{Y}}{2} + \frac{s \,.\, n}{2} \frac{1}{\cos^2\frac{\mathrm{X}-\mathrm{Y}}{2}} = 0,$$

which is integrable; and its integral is $\frac{ds}{d\mathrm{Y}} + n \,.\, s \tan\frac{\mathrm{X}-\mathrm{Y}}{2}$ = some arbitrary function of Y, as χ Y. Integrate again, then

$$s = \varepsilon^{-\int n \tan\frac{\mathrm{X}-\mathrm{Y}}{2} d\mathrm{Y}} \left(\int \varepsilon^{\int n \tan\frac{\mathrm{X}-\mathrm{Y}}{2} d\mathrm{Y}} \chi \mathrm{Y}\, d\mathrm{Y} + \psi \mathrm{X}\right),$$

where ψ is arbitrary. Effecting these integrations, and reducing,

$$s = \cos^{-2n}\frac{\mathrm{X}-\mathrm{Y}}{2}\left(\int \cos^{2n}\frac{\mathrm{X}-\mathrm{Y}}{2} \,.\, \chi \mathrm{Y}\, d\mathrm{Y} + \psi \mathrm{X}\right).$$

To return to P_n, we have the following systems of equations:

$\mathrm{P}_n = \int v\, d\mathrm{Y}$,

$$v = \varepsilon^{-\int \tan\frac{\mathrm{X}-\mathrm{Y}}{2} d\mathrm{Y}} \left(\int t \varepsilon^{\int \tan\frac{\mathrm{X}-\mathrm{Y}}{2} d\mathrm{Y}} d\mathrm{Y}\right) = \cos^{-2}\frac{\mathrm{Y}-\mathrm{X}}{2} \int t \cos^2\frac{\mathrm{Y}-\mathrm{X}}{2} d\mathrm{Y},$$

$$t = \cos^{-4}\frac{\mathrm{Y}-\mathrm{X}}{2} \int q \cos^4\frac{\mathrm{Y}-\mathrm{X}}{2} \,.\, d\mathrm{Y},$$

$$\vdots$$

$$\varrho = \cos^{-2(n-1)}\frac{\mathrm{Y}-\mathrm{X}}{2} \int s \cos^{2(n-1)}\frac{\mathrm{Y}-\mathrm{X}}{2} \,.\, d\mathrm{Y},$$

$$s = \cos^{-2n}\frac{\mathrm{Y}-\mathrm{X}}{2}\left(\int \cos^{2n}\frac{\mathrm{Y}-\mathrm{X}}{2} \,.\chi \mathrm{Y}\, d\mathrm{Y} + \psi \mathrm{X}\right);$$

whence

$$\mathrm{P}_n = \ldots\ldots \int \cos^{-2}\frac{\mathrm{Y}-\mathrm{X}}{2} \int \cos^{-2}\frac{\mathrm{Y}-\mathrm{X}}{2} \left(\int \cos^{2n}\frac{\mathrm{X}-\mathrm{Y}}{2} \chi \mathrm{Y}\, d\mathrm{Y} + \psi \mathrm{X}\right) d\mathrm{Y}\, d\mathrm{Y} \ldots (n \text{ times.})$$

Now $\cos\frac{\mathrm{Y}-\mathrm{X}}{2} = \cos\left(\sqrt{-1}\log\sqrt{\frac{1+\mu}{1-\mu}}\right) = \frac{1}{2}\left(\sqrt{\frac{1+\mu}{1-\mu}} + \sqrt{\frac{1-\mu}{1+\mu}}\right)$,

and $\cos^2\frac{\mathrm{Y}-\mathrm{X}}{2} = \frac{1}{1-\mu^2}$; and the complete integral will be expressed, by substituting for X and Y in terms of μ and φ.

6. But an important point yet remains to be determined. The original equation, being a partial differential equation of the second order, can only involve in its integral two arbitrary functions. But here, after χ Y and ψ X have come in by two integrations, we have n integrations to perform with respect to Y. It would seem, therefore, that no constant or arbitrary function of X must be added in these integrations.

Such is not the case. At each integration a function of X must be added, and these functions determined by reference to the original differential equation*.

7. Returning to the value of P_n, we have

$$P_0 = \chi_1\left(\varphi - \tfrac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu}\right) + \psi\left(\varphi + \tfrac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu}\right).$$

Now in the calculation of attractions, where P_0 is the coefficient of r^0 in the expansion of

$$\left\{r^2 + r'^2 - rr'\left(\mu\mu' + \sqrt{(1-\mu^2)}\sqrt{(1-\mu'^2)}\cos(\varphi-\varphi')\right)\right\}^{-\frac{1}{2}},$$

we know that it is 1; consequently

$$\chi_1\left(\varphi - \tfrac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu}\right) + \psi\left(\varphi + \tfrac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu}\right) = 1,$$

and expanding by TAYLOR's theorem, we get

$$\left.\begin{aligned}&\chi_1\varphi - \chi'_1\varphi\,\tfrac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu} - \frac{\chi''_1\varphi}{2}\left(\tfrac{1}{2}\log\frac{1+\mu}{1-\mu}\right)^2\\ &\quad + \frac{\chi'''_1\varphi}{2.3}\sqrt{-1}\left(\tfrac{1}{2}\log\frac{1+\mu}{1-\mu}\right)^3 + \&c.\\ &+\psi\varphi + \psi'\varphi\,\tfrac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu} - \frac{\psi''\varphi}{2}\left(\tfrac{1}{2}\log\frac{1+\mu}{1-\mu}\right)^2\\ &\quad - \frac{\psi'''\varphi}{2.3}\sqrt{-1}\left(\tfrac{1}{2}\log\frac{1+\mu}{1-\mu}\right)^3 + \&c.\end{aligned}\right\} = 1.$$

By equating the coefficients of the same powers of $\frac{1}{2}\log\frac{1+\mu}{1-\mu}$, we have $\chi_1\varphi + \psi\varphi = 1$, and $\psi'\varphi - \chi'_1\varphi = 0$, or $\psi\varphi - \chi_1\varphi =$ constant.

Therefore $\psi\varphi$ and $\chi_1\varphi$ are absolute constants, and their sum is 1; whence it follows that $\chi\varphi = 0$. Let $\psi\phi = \frac{c}{2}$, then

$$P_n = c\left(\ldots\int\cos^{-2}\frac{Y-X}{2}\int\cos^{-2}\frac{Y-X}{2}\int\frac{1}{2}\cos^{-2}\frac{Y-X}{2}\,dY\,dY\ldots(n\text{ times})\right).$$

Effecting these integrations, we find that P_n consists of a series of powers of

* The common differential equation $(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + n(n+1)P_n = 0$ will illustrate this point. Let $P_n = \frac{d^{-n}z}{d\mu^{-n}}$, and after substitution, differentiate n times; then $(1-\mu^2)\frac{d^2z}{d\mu^2} - 2(n+1)\mu\frac{dz}{d\mu} = 0$, whence $z = \int\frac{k\,d\mu}{(1-\mu^2)^{n+1}} + m$. It is clear that no more *arbitrary* constants than k and m can be introduced; and yet if the integrals were left indefinite, we might obtain an integral of an expression which should differ from the integral of the same expression obtained by a slightly different process, by a constant. By another integration this would cease to be a constant, and we should obtain thus different values for P_n. The fact is, that constants must be added at each integration, and recourse had to the original equation, to determine them in terms of m, k, and μ.

$\tan \frac{Y-X}{2}$, whose coefficients may, for anything we yet know, be functions of X. The following process shows that constants as coefficients will satisfy the original equation, and determines them. The integration itself gives the coefficient of the highest power.

Let then*

$$P_n = c\left\{\begin{array}{l} \frac{2^{n-1}}{[n]} \tan^n \frac{Y-X}{2} + c' \tan^{n-1} \frac{Y-X}{2} + c'' \tan^{n-2} \frac{Y-X}{2} \\ + \ldots + c^{(n-2)} \tan^2 \frac{Y-X}{2} + c^{(n-1)} \tan \frac{Y-X}{2} + c^{(n)} \end{array}\right\}.$$

Then

$$\frac{dP_n}{dY} = c\left\{\begin{array}{l} \frac{2^{n-2}}{[n-1]} \tan^{n-1} \frac{Y-X}{2} + \frac{n-1}{2} c' \tan^{n-2} \frac{Y-X}{2} \\ + \frac{n-2}{2} c'' \tan^{n-3} \frac{Y-X}{2} + \ldots + \frac{2}{2} c^{(n-2)} \tan \frac{Y-X}{2} \\ + \frac{1}{2} c^{(n-1)} \end{array}\right\}\left(1 + \tan^2 \frac{Y-X}{2}\right),$$

$$= c\left\{\begin{array}{l} \frac{2^{n-2}}{[n-1]} \tan^{n+1} \frac{Y-X}{2} + \frac{n-1}{2} c' \tan^n \frac{Y-X}{2} + \left(\frac{n-2}{2} c'' + \frac{2^{n-2}}{[n-1]}\right) \tan^{n-1} \frac{Y-X}{2} \\ + \left(\frac{n-3}{2} c''' + \frac{n-1}{2} c'\right) \tan^{n-2} \frac{Y-X}{2} \\ + \left(\frac{n-4}{2} c^{\text{iv}} + \frac{n-2}{2} c''\right) \tan^{n-3} \frac{Y-X}{2} + \ldots + \left(\frac{3}{2} c^{(n-3)} + \frac{5}{2} c^{(n-5)}\right) \tan^4 \frac{Y-X}{2} \\ + \left(\frac{2}{2} c^{(n-2)} + \frac{4}{2} c^{(n-4)}\right) \tan^3 \frac{Y-X}{2} \\ + \left(\frac{1}{2} c^{(n-1)} + \frac{3}{2} c^{(n-3)}\right) \tan^2 \frac{Y-X}{2} + \frac{2}{2} c^{(n-2)} \tan \frac{Y-X}{2} + \frac{1}{2} c^{(n-1)} \end{array}\right\}.$$

$$\frac{d^2 P_n}{dX\,dY} = -\frac{c}{\cos^2 \frac{Y-X}{2}}\left\{\begin{array}{l} \frac{(n+1)\,2^{n-3}}{[n-1]} \tan^n \frac{Y-X}{2} + \frac{n(n+1)}{4} c' \tan^{n-1} \frac{Y-X}{2} \\ + \frac{n-1}{2}\left(\frac{n-2}{2} c'' + \frac{2^{n-2}}{[n-1]}\right) \tan^{n-2} \frac{Y-X}{2} \\ + \frac{n-2}{2}\left(\frac{n-3}{2} c''' + \frac{n-1}{2} c'\right) \tan^{n-3} \frac{Y-X}{2} \\ + \ldots + \frac{4}{2}\left(\frac{3}{2} c^{(n-3)} + \frac{5}{2} c^{(n-5)}\right) \tan^3 \frac{Y-X}{2} \\ + \frac{3}{2}\left(\frac{2}{2} c^{(n-2)} + \frac{4}{2} c^{(n-4)}\right) \tan^2 \frac{Y-X}{2} \\ + \frac{2}{2}\left(\frac{1}{2} c^{(n-1)} + \frac{3}{2} c^{(n-3)}\right) \tan \frac{Y-X}{2} - \frac{1}{2}\frac{2}{2} c^{(n-2)} \end{array}\right\}.$$

* $[n] = 1.2.3.4\ldots.n$.

$$n\frac{n+1}{4}\mathrm{P}_n\frac{1}{\cos^2\frac{\mathrm{Y}-\mathrm{X}}{2}} = \frac{c}{\cos^2\frac{\mathrm{Y}-\mathrm{X}}{2}}\left\{\begin{array}{l}\frac{(n+1)2^{n-3}}{[n-1]}\tan^n\frac{\mathrm{Y}-\mathrm{X}}{2}+\frac{n(n+1)}{4}c'\tan^{n-1}\frac{\mathrm{Y}-\mathrm{X}}{2}\\ +\frac{n(n+1)}{4}c''\tan^{n-2}\frac{\mathrm{Y}-\mathrm{X}}{2}+\ldots.\\ \ldots+\frac{n(n+1)}{4}c^{(n-1)}\tan\frac{\mathrm{Y}-\mathrm{X}}{2}+\frac{n(n+1)}{4}c^{(n)}\end{array}\right\}.$$

Whence we obtain $c'=0$, $c'''=0$, $c^{\mathrm{v}}=0$, &c.

Also

$$\left(\frac{n(n+1)}{4}-\frac{(n-1)(n-2)}{4}\right)c''=\frac{2^{n-3}}{[n-2]};\ \left(\frac{n(n+1)}{4}-\frac{(n-3)(n-4)}{4}\right)c^{\mathrm{iv}}=\frac{(n-3)(n-2)}{4}c'';$$

$$\left(\frac{n(n+1)}{4}-\frac{(n-5)(n-6)}{4}\right)c^{\mathrm{vi}}=\frac{(n-5)(n-4)}{4}c^{\mathrm{iv}};\ \text{\&c. \&c.}$$

Consequently

$$\mathrm{P}_n = c\left\{\begin{array}{l}\frac{2^{n-1}}{[n]}\tan^n\frac{\mathrm{Y}-\mathrm{X}}{2}+\frac{2^{n-1}}{2[n-2](2n-1)}\tan^{n-2}\frac{\mathrm{Y}-\mathrm{X}}{2}\\ +\frac{2^{n-1}}{2.4.[n-4](2n-1)(2n-3)}\tan^{n-4}\frac{\mathrm{Y}-\mathrm{X}}{2}\\ +\frac{2^{n-1}\tan^{n-6}\frac{\mathrm{Y}-\mathrm{X}}{2}}{2.4.6.[n-6].(2n-1)(2n-3).(2n-5)}+\ldots\end{array}\right\}.$$

Now

$$\frac{\mathrm{Y}-\mathrm{X}}{2}=-\frac{1}{2}\sqrt{-1}\log\frac{1+\mu}{1-\mu}=\tan^{-1}\left(-\mu\sqrt{-1}\right),\ldots\tan\frac{\mathrm{Y}-\mathrm{X}}{2}$$
$$=\left(-\mu\sqrt{-1}\right);$$

whence, finally,

$$\mathrm{P}_n=2^{n-1}c\left(\frac{(-\mu\sqrt{-1})^n}{[n]}+\frac{(-\mu\sqrt{-1})^{n-2}}{2.[n-2].(2n-1)}+\frac{(-\mu\sqrt{-1})^{n-4}}{2.4.[n-4](2n-1)(2n-3)}+\ldots\right),$$

a remarkable result, showing that in this instance P_n is independent of ϕ.

P_n being free from ϕ, is a perfectly symmetrical function of μ and μ'; and μ' is a constant with respect to μ; therefore

$$\mathrm{P}_n=\mathrm{K}_n\left(\frac{(-\mu\sqrt{-1})^n}{[n]}+\frac{(-\mu\sqrt{-1})^{n-2}}{2[n-2](2n-1)}+\frac{(-\mu\sqrt{-1})^{n-4}}{2.4.[n-4](2n-1)(2n-3)}+\ldots\right)\times$$
$$\left(\frac{-\mu'\sqrt{-1})^n}{[n]}+\frac{(-\mu'\sqrt{-1})^{n-2}}{2.[n-2](2n-1)}+\ldots\right).$$

To determine K_n for any particular value of n, we refer to the expression from which the two series were deduced; namely,

$$\left\{r^2+r'^2-2rr'\left(\mu\mu'+\sqrt{(1-\mu^2)}\sqrt{(1-\mu'^2)}\cos(\phi-\phi')\right)\right\}^{-\frac{1}{2}}.$$

When μ and μ' are each 1, then $\mathrm{P}_n=1$, which gives an equation to find K_n.

8. Returning to the theory of attractions, we have, when the particle is internal,

$$V = \int_r^R \int_{-1}^1 \int_0^{2\pi} \varrho\, r' \left(P_0 + P_1 \frac{r}{r'} + \ldots + P_n \frac{r^n}{r'^n} + \ldots\right) d r'\, d \mu'\, d \varphi'$$

$$= 2\pi \int_r^R \int_{-1}^1 \varrho \left(P_0 r' + P_1 r + P_2 \frac{r^2}{r'} + \ldots + P_n \frac{r^n}{r'^{n-1}} + \ldots\right) d r'\, d \mu',$$

for that portion of the body which is comprised between a sphere of radius r, and the surface of the body, supposed to be a surface of revolution round the axis of z; for, in that case R, the value of r' at the surface, is independent of φ'.

9. Suppose, for example, we wish to find the attraction of a homogeneous spheroid, on a point within it. In this case ϱ is constant, and

$$R^2 = \left(\frac{1-\mu'^2}{a^2} + \frac{\mu'^2}{c^2}\right)^{-1};$$

a being the semi-major, and c the semi-minor axis.

First, all the even terms vanish; for the general even term is

$$2\pi\varrho \int_r^R \int_{-1}^1 P_{2n+1} \frac{r^{2n+1}}{r'^{2n}}\, d r'\, d \mu' = -\frac{2\pi\varrho\, r^{2n+1}}{2n-1} \int_{-1}^1 P_{2n+1} \left(\frac{1-\mu'^2}{a^2} + \frac{\mu'^2}{c^2}\right)^{\frac{2n-1}{2}} d\mu'$$

$$+ \frac{2\pi\varrho\, r^2}{2n-1} \int_{-1}^1 P_{2n+1}\, d\mu'.$$

Now P_{2n+1} consists of odd powers of μ'; and $\left(\frac{1-\mu'^2}{a^2} + \frac{\mu'^2}{c^2}\right)^{\frac{2n-1}{2}}$ can be expanded in even powers of μ'; therefore the integral of the product (which is an odd function), taken from $\mu' = -1$ to $\mu' = 1$, is 0. Also $\int_{-1}^1 P_{2n+1}\, d\mu' = 0$ *. All the odd terms above the third vanish; for the $(2n+1)$th term is

$$2\pi\varrho \int_r^R \int_{-1}^1 P_{2n} \frac{r^{2n}}{r'^{2n-1}}\, d r'\, d\mu' = -\frac{\pi\varrho}{n-1} \int_{-1}^1 P_{2n}\, r^{2n} \left(\frac{1-\mu'^2}{a^2} + \frac{\mu'^2}{c^2}\right)^{n-1} d\mu'$$

$+ \frac{\pi\rho}{n-1} \int_{-1}^1 P_{2n}\, r^2\, d\mu'$, for it may be shown that $\int_{-1}^1 \int_0^{2\pi} P_\iota P_{\iota'}\, d\mu'\, d\varphi' = 0$, if ι and ι' be different integers. Now when n is greater than 1, $\left(\frac{1-\mu'^2}{a^2} + \frac{\mu'^2}{c^2}\right)^{n-1}$ is a rational and entire function of μ', and, therefore, capable of being expressed in a series of LAPLACE's coefficients †, the highest of which will be of the $(2n-2)$th order; and therefore no term of this expansion can be of the same order as P_{2n}; and the integral of the product of any two of different orders, between these limits, vanishes. So the second member of this vanishes.

* See Pratt. Mec. Phil., § 180.

† See Pratt. Mec. Phil., § 176. POISSON, Théorie Math. de la Chal., chap. viii. LAPLACE, Méc. Cél. liv. iii. chap. ii.

The first term is

$$2\pi\varrho\int_r^R\int_{-1}^1 r'\,dr'\,d\mu' = \pi\varrho\int_{-1}^1\left(\frac{1-\mu'^2}{a^2}+\frac{\mu'^2}{c^2}\right)^{-1}d\mu' - \pi\varrho r^2\int_{-1}^1 d\mu'$$

$$= 2\pi\varrho\left(a^2\frac{\sqrt{1-e^2}}{e}-r^2\right), \text{ (e being the eccentricity).}$$

The third term is

$$\frac{\pi}{2}K_2\varrho\int_r^R\int_{-1}^1\left(\frac{1}{3}-\mu^2\right)\left(\frac{1}{3}-\mu'^2\right)\frac{r^2}{r'}\,dr'\,d\mu'$$

$$= -\frac{\pi\varrho}{4}K_2\int_{-1}^1\left(\frac{1}{3}-\mu^2\right)\left(\frac{1}{3}-\mu'^2\right)r^2\log\left(\frac{1-\mu'^2}{a^2}+\frac{\mu'^2}{c^2}\right)d\mu',$$

$$= -\pi\varrho\frac{K_2}{2}\left(\frac{1}{3}-\mu^2\right)r^2\left(\frac{2}{3}\frac{c\,a^2}{(a^2-c^2)^{\frac{3}{2}}}\tan^{-1}\frac{\sqrt{(a^2-c^2)}}{c}-\frac{2}{3}\frac{c^2}{a^2-c^2}-\frac{4}{9}\right),$$

$$= -3\pi\varrho\left(\frac{1}{3}-\mu^2\right)r^2\left(\frac{1}{3}-\frac{1}{e^2}+\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right), \text{ for } K_2=9.$$

The value of V for the sphere of radius r, calculated by the usual method, is $\frac{4\pi\rho r^2}{3}$*; consequently, for the whole ellipsoid, the value of V is

$$\frac{4\pi\rho r^2}{3}+2\pi\varrho\left(a^2\frac{\sqrt{(1-e^2)}}{e}\sin^{-1}e-r^2\right)-3\pi\varrho\left(\frac{1}{3}-\mu^2\right)r^2\left(\frac{1}{3}-\frac{1}{e^2}+\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right).$$

Differentiate to f, by means of the equations $r^2=f^2+g^2+h^2$ and $\mu=\frac{h}{r}$, and we have

$$\left.\begin{aligned}
-\frac{dV}{df} &= \text{attraction in } x = +2\pi\varrho f\left(1-\frac{1}{e^2}+\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right);\\
\text{so } -\frac{dV}{dg} &= \text{attraction in } y = +2\pi\varrho g\left(1-\frac{1}{e^2}+\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right);\\
\text{so } -\frac{dV}{dh} &= \text{attraction in } y = +4\pi\varrho h\left(+\frac{1}{e^2}-\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right);
\end{aligned}\right\}$$

which are the common expressions otherwise found†.

Also

$$-\frac{dV}{dr} = \text{attraction to centre} = +4\pi\varrho\left\{\frac{r}{3}+\frac{3}{2}r\left(\frac{1}{3}-\mu^2\right)\left(\frac{1}{3}-\frac{1}{e^2}+\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right)\right\}.$$

10. By a similar process, I have deduced the attraction of an oblate spheroid, on a point within it; the density varying inversely as the distance from the centre. The corresponding expressions are

$$-\frac{dV}{df} = +2\pi\varrho\frac{f}{r}-\frac{3}{4}\frac{\pi\varrho f}{a(1-e^2)}\left\{\frac{1}{e^2}-\frac{1}{3}-\left(\frac{(1-e^2)^2}{2e^3}+\frac{2}{3}\frac{1-e^2}{e}\right)\log\frac{1+e}{1-e}\right\};$$

$$-\frac{dV}{dg} = +2\pi\varrho\frac{g}{r}-\frac{3}{4}\frac{\pi\varrho g}{a(1-e^2)}\left\{\frac{1}{e^2}-\frac{1}{3}-\left(\frac{(1-e^2)^2}{2e^3}+\frac{2}{3}\frac{1-e^2}{e}\right)\log\frac{1+e}{1-e}\right\};$$

* See Pratt. Mec. Phil., § 172.

† Ibid. § 158.

$$-\frac{dV}{dh} = +2\pi\varrho\frac{h}{r} + \frac{3}{2}\frac{\pi\varrho h}{a(1-e^2)}\left\{\frac{1}{e^2} - \frac{1}{3} - \left(\frac{(1-e^2)^2}{2e^3} + \frac{2}{3}\frac{1-e^2}{e}\right)\log\frac{1+e}{1-e}\right\};$$

$$-\frac{dV}{dr} = +2\pi\varrho + \frac{3}{4}\frac{\pi\varrho(3\mu^2-1)}{a(1-e^2)}\left\{\frac{1}{e^2} - \frac{1}{3} - \left(\frac{(1-e^2)^2}{2e^3} + \frac{2}{3}\frac{1-e^2}{e}\right)\log\frac{1+e}{1-e}\right\}.$$

11. When the particle attracted is external, the series in (3.) does not give a finite expression. Instead of taking it separately, I make it a case of a general theorem which follows.

12. To find the attraction of a spheroid on a point within it, the density being any function of the distance from the centre, and the eccentricity being small.

Let $\varrho\,\phi\,r'$ represent the law of density; then the value of V, for the portion comprised between the surface and a sphere of radius r, is

$$2\pi\varrho\int_r^R\int_{-1}^1 \phi r' . r'\left(P_0 + P_1\frac{r}{r'} + \ldots + P_n\frac{r^n}{r'^n} + \ldots\right)dr'\,d\mu'.$$

Integrating by parts we have

$$\int\phi r' . r'^{1-n} . dr' = \frac{\phi_{,} r'}{r'^{n-1}} + (n-1)\frac{\phi_{,,} r'}{r'^n} + (n-1)n\frac{\phi_{,,,} r'}{r'^{n+1}} + (n-1)n(n+1)\frac{\phi_{iv} r'}{r'^{n+1}} + \&c.$$

Therefore the $(n+1)$th term of V is

$$2\pi\varrho r^n\int_{-1}^1 P_n\left(\frac{\phi_{,}R}{R^{n-1}} + (n-1)\frac{\phi_{,,}R}{R^n} + (n-1)n\frac{\phi_{,,,}R}{R^{n+1}} + (n-1)n(n+1)\frac{\phi_{iv}R}{R^{n+2}} + \ldots\right)d\mu'$$

$$-2\pi\varrho r^n\int_{-1}^1 P_n\left(\frac{\phi_{,}r}{r^{n-1}} + (n-1)\frac{\phi_{,,}r}{r^n} + (n-1)n\frac{\phi_{,,,}r}{r^{n+1}} + (n-1)n(n+1)\frac{\phi_{iv}r}{r^{n+2}} + \ldots\right)d\mu'.$$

Now $R = \left\{\frac{1}{a^2} + \left(\frac{1}{c^2} - \frac{1}{a^2}\right)\mu'^2\right\}^{-\frac{1}{2}} = a(1+e^2\mu'^2)^{-\frac{1}{2}} = a\left(1 - \frac{e^2}{2}\mu'^2\right)$, rejecting e^4 and higher powers of e; and $\phi_{(m)}R = \phi_{(m)}a - \phi_{(m-1)}a . \frac{ae^2\mu'^2}{2}$ to the same degree of accuracy. The last member of the expression for V need only be calculated when $n = 0$; for all the rest of the terms $\left(\text{involving} \int_{-1}^1 P_n\,d\mu' \text{ where } n > 0\right)$ vanish.

The first member need only be calculated when $n = 0$, and when $n = 2$; for when n is odd, it vanishes as before; and also when n is even and greater than 2: for the functions of R involve no higher powers of μ' than the square; and consequently they vanish, when multiplied by P_4, P_6, .. &c., and integrated with respect to μ', from -1 to $+1$*.

When $n = 0$, the term is

$$2\pi\varrho\int_{-1}^1(R\phi_{,}R - \phi_{,,}R)\,d\mu' - 2\pi\varrho\int_{-1}^1(r\phi_{,}r - \phi_{,,}r)\,d\mu',$$

$$= 2\pi\varrho\int_{-1}^1\left(a\phi_{,}a - \phi a . \frac{a^2e^2\mu'^2}{2} - \phi_{,,}a\right)d\mu' - 4\pi\varrho(r\phi_{,}r - \phi_{,,}r),$$

$$= 4\pi\varrho\left(a\phi_{,}a - \phi_{,,}a - \frac{e^2}{6}a^2\phi a - (r\phi_{,}r - \phi_{,,}r)\right).$$

* See Art. 9.

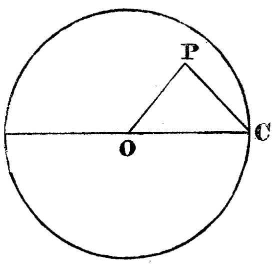
When $n = 2$, the term is

$$\frac{9}{2}\pi \varrho r^2\left(\mu^2 - \frac{1}{3}\right)\int_{-1}^{1}\left(\mu'^2 - \frac{1}{3}\right)\left(\frac{\phi_{,} R}{R} + \frac{\phi_{,,} R}{R^2} + 2\frac{\phi_{,,,} R}{R^3} + 2\,.\,3\frac{\phi_{iv} R}{R^4} + \ldots\right) d\mu'$$

$$= \frac{9}{2}\pi \varrho r^2\left(\mu^2 - \frac{1}{3}\right)\Sigma_1^\infty[m-1]\int_{-1}^{1}\left(\mu'^2 - \frac{1}{3}\right)\frac{\phi_m R}{R^m}\, d\mu',$$

$$= \frac{9}{2}\pi \varrho r^2\left(\mu^2 - \frac{1}{3}\right)\Sigma_1^\infty[m-1]\int_{-1}^{1}\left(\mu'^2 - \frac{1}{3}\right)\left(\frac{\phi_m a}{a^m} - \frac{\phi_{m-1} a}{a^{m-1}}\,.\,\frac{e^2\mu'^2}{2} + \frac{\phi_m a}{a^m}\,.\,\frac{m e^2 \mu'^2}{2}\right) d\mu',$$

$$= \frac{2}{5}\pi \varrho r^2\left(\mu^2 - \frac{1}{3}\right) e^2\, \Sigma_1^\infty\,[m-1]\left(\frac{\phi_m a}{a^m}\,.\,m - \frac{\phi_{m-1} a}{a^{m-1}}\right) = -\frac{2}{5}\pi \varrho r^2\left(\mu^2 - \frac{1}{3}\right) e^2 \phi\, a.$$



To calculate V for the sphere whose radius is r.

Let the sphere be referred to polar coordinates, the centre being the pole.

Let $O C = r$, $O P = r_1$, and $POC = \theta$; then $P C = \sqrt{(r^2 + r_1^2 - 2 r r_1 \cos\theta)}$.

Mass of the element at $P = \varrho\, r_1^2\, \phi\, r_1\, d r_1 \sin\theta\, d\theta\, d\omega$, and

$$V = \int_0^r\int_0^\pi\int_0^{2\pi}\frac{\rho\, r_1^2\, \phi\, r_1\, d r_1 \sin\theta\, d\theta\, d\omega}{\sqrt{(r^2 + r_1^2 - 2 r r_1\cos\theta)}} = 2\pi\varrho\int_0^r\int_0^\pi\frac{r_1^2\,\phi\, r_1 \sin\theta\, d r_1\, d\theta}{\sqrt{(r^2 + r_1^2 - 2 r r_1 \cos\theta)}}$$

$$= 2\pi\varrho\int_0^r \frac{r_1\,\phi_1 r_1}{r}\Big((r + r_1) - (r - r_1)\Big)\, d r_1 = \frac{4\pi\rho}{r}\int_0^r r_1^2\,\phi\, r_1\, d r_1$$

$$= \frac{4\pi\rho}{r}\left(r^2\,\phi_{,} r - 2 r\,\phi_{,,} r + 2\,\phi_{,,,} r - K\right),$$

K being the value of $r_1^2\,\phi_1 r_1 - 2 r_1\,\phi_{,,} r_1 + 2\,\phi_{,,,} r_1$, when $r_1 = 0$.

The whole value of V then is

$$4\pi\varrho\left\{a\,\phi_{,} a - \phi_{,,} a - \frac{1}{6} e^2 a^2 \phi\, a - \left(\phi_{,,} r - \frac{2}{r}\phi_{,,,} r + \frac{K}{r}\right) - \frac{1}{10} r^2\left(\mu^2 - \frac{1}{3}\right) e^2 \phi\, a\right\}.$$

And the attraction toward the centre

$$= -\frac{dV}{dr} = 4\pi\varrho\left\{\frac{\psi r}{r^2} + \frac{1}{5}\left(\mu^2 - \frac{1}{3}\right) r\, e^2\, \phi\, a\right\},$$

where

$$\psi r = 2\,\phi_{,,,} r - 2 r\,\phi_{,,} r + r^2\,\phi_{,} r - K = \int_0^r r^2\,\phi\, r\, dr.$$

13. To find the attraction of the same spheroid on a particle without it.

The series (Art. 3.) is

$$2\pi\varrho\int_0^R\int_{-1}^{1}\phi\, r'\left(P_0\frac{r'^2}{r} + P_1\frac{r'^3}{r^2}\,.+\,.\,.\right) d r'\, d\mu'.$$

Now

$$\int \phi\, r'\,.\, r'^{n+1}\,.\, d r' = r'^{n+1}\,\phi_{,} r' - (n+1)\, r'^n\,\phi_{,,} r' + n(n+1)\, r'^{n-1}\,\phi_{,,,} r' - \&c. \text{ (A)};$$

and the general (nth) term of V is

$$\frac{2\pi\rho}{r^n}\int_{-1}^{1} P_{n-1}\left(r'^{n+1}\,\phi_{,} r' - (n+1)\, r'^n\,\phi_{,,} r' + n(n+1)\, r'^{n-1}\,\phi_{,,,} r' - \&c.\right) d\mu'$$

$$-\frac{2\pi\rho}{r^n}\int_{-1}^{1} P_{n-1}\, C\, d\mu',$$

where C is the value of (A) when $r' = 0$. As before, the last term need be calculated only when $n = 1$; and the first when $n = 1$, and $n = 3$.

When $n = 1$, it is

$$\frac{2\pi\rho}{r}\int_{-1}^{1}(\mathrm{R}^2\,\phi_{\prime}\,\mathrm{R} - 2\,\mathrm{R}\,\phi_{\prime\prime}\,\mathrm{R} + 2\,\phi_{\prime\prime\prime}\,r - \mathrm{K})\,d\,\mu' = \frac{2\pi\rho}{r}\int_{-1}^{1}\psi\,r\,d\,\mu'$$

$$= \frac{4\pi\rho}{r}\left(\psi\,a - \frac{e^2}{6}\,a^3\,\phi\,a\right).$$

When $n = 3$, it is

$$\frac{9}{2}\left(\mu^2 - \frac{1}{3}\right)\frac{\pi\rho}{r^3}\int_{-1}^{1}d\,\mu'\left(\mu'^2 - \frac{1}{3}\right)\Big(\mathrm{R}^4\,\phi_{\prime}\,\mathrm{R} - 4\,\mathrm{R}^3\,\phi_{\prime\prime}\,\mathrm{R} + 12\,\mathrm{R}^2\,\phi_{\prime\prime\prime}\,\mathrm{R} - 24\,\mathrm{R}\,\phi_{\mathrm{iv}}\,\mathrm{R}$$

$$+ 24\,\phi_{\mathrm{v}}\,\mathrm{R}\Big) = -\frac{2}{5}\left(\mu^2 - \frac{1}{3}\right)\frac{\pi\rho\,a^5}{r^3}\,\phi\,a\,.\,e^2.$$

The whole value of V is

$$\frac{4\pi\rho}{r}\left\{\psi\,a - \frac{e^2}{6}\,a^3\,\phi\,a - \frac{1}{10}\left(\mu^2 - \frac{1}{3}\right)\frac{a^5}{r^2}\,\phi\,a\,.\,e^2\right\}.$$

And

$$-\frac{d\,\mathrm{V}}{d\,r} = \frac{4\pi\rho}{r^2}\left\{\psi\,a - \frac{e^2}{6}\,a^3\,\phi\,a - \frac{3}{10}\left(\mu^2 - \frac{1}{3}\right)\frac{a^5}{r^2}\,\phi\,a\,.\,e^2\right\}.$$

14. Instead of the eccentricity $e = \sqrt{\left(1 - \frac{c^2}{a^2}\right)}$ it will be more convenient to employ the ellipticity $\varepsilon = 1 - \frac{c}{a}$*. These give $\varepsilon = \frac{e^2}{2}$. And the values of V become for an internal point,

$$4\pi\varrho\left\{a\,\phi_{\prime}\,a - \phi_{\prime\prime}\,a - \left(\phi_{\prime\prime}\,r - \frac{2}{r}\,\phi_{\prime\prime\prime}\,r + \frac{\mathrm{K}}{r}\right) - \frac{\varepsilon}{3}\,a^2\,\phi\,a - \frac{r^2}{5}\left(\mu^2 - \frac{1}{3}\right)\varepsilon\,\phi\,a\right\};$$

for an external point,

$$4\,\pi\,\varrho\,\frac{1}{r}\left\{\psi\,a - \frac{\varepsilon}{3}\,a^3\,\phi\,a - \frac{3}{5}\,\frac{1}{r^2}\left(\mu^2 - \frac{1}{3}\right)a^5\,\phi\,a\right\}.$$

15. To find the attraction on the supposition that the body is composed of spheroidal layers, homogeneous in themselves, but differing from one another in density and ellipticity.

First, on an internal point.

Let r', as before, be the radius vector of any layer; a' its equatorial radius; $\varrho\,\phi\,a'$ its density, and ε' its ellipticity, being some function of a' as $\chi\,a'$. Then

$$a' = r'\,(1 + \varepsilon'\,\mu'^2) \text{ and } \phi\,a' = \phi\,r' + r'\,\phi'\,r'\,\chi\,r'\,\mu'^2 = \phi\,r' + \mathrm{F}\,r'\,.\,\mu'^2, \text{ suppose.}$$

Consequently to the term before produced in (12.) by $\phi\,r'$ we must add a term similarly produced by $\mathrm{F}\,r'\,.\,\mu'^2$. Also, instead of taking, in the first instance, the portion comprised between the surface and a sphere of radius r, we must take the por-

* See PUISSANT, vol. i. p. 259, where the word ellipticity is used in this sense.

tion between the surface and that spheroid on which the point lies, whose ellipticity is ε_1. The $(n+1)$th term of V now becomes

$$2\pi\varrho r^n \int_{-1}^{1} \mathrm{P}_n \, d\mu' \left\{ \int_r^{\mathrm{R}} \frac{\Phi r'}{r'^{n-1}} \, d r' + \mu'^2 \int_r^{\mathrm{R}} \frac{\mathrm{F} r'}{r'^{n-1}} \, d r' \right\}.$$

The first part of this gives $4\pi\varrho \left\{ \begin{array}{l} a\varphi_{,}a - \varphi_{,,}a - \frac{\varepsilon}{3} a^2 \varphi a - \frac{\varepsilon}{5}\left(\mu^2 - \frac{1}{3}\right) r^2 \varphi a \\ - \left(\mathrm{a}\varphi_{,}\mathrm{a} - \varphi_{,,}\mathrm{a} - \frac{\varepsilon_1}{3}\mathrm{a}^2 \varphi \mathrm{a} - \frac{\varepsilon_1}{5}\left(\mu^2 - \frac{1}{3}\right) r^2 \varphi \mathrm{a}\right) \end{array} \right\}$,

a being semiaxis major of the stratum on which r lies. To determine the other part, it is necessary to compute it when $n = 0$ and $n = 2$, which gives

$$4\pi\varrho \left\{ \frac{1}{3} \int_{\mathrm{a}}^{a} a' \, \mathrm{F} a' \, d a' + \frac{1}{5} r^2 \left(\mu^2 - \frac{1}{3}\right) \int_{\mathrm{a}}^{a} \frac{\mathrm{F} a'}{a'} \, d a' \right\}.$$

To the sum of these we must add the value of V for the inner spheroid; and for this purpose we have to obtain V for an external point.

To the expression in (13.) we must add

$$\frac{2\pi\varrho}{r^n} \int_{-1}^{1} d\mu' \left(\mu'^2 \mathrm{P}_{n-1} \int_0^{\mathrm{R}} \mathrm{F} r' \, r'^{n+1} \, d r'\right),$$

to be calculated when $n = 1$ and $n = 3$. This is

$$4\pi\varrho \left\{ \frac{1}{3r} \int_0^{a} \mathrm{F} a' \,.\, a'^2 \, d a' + \frac{\mu^2 - \frac{1}{3}}{5 r^3} \int_0^{a} \mathrm{F} a' \,.\, a'^4 \, d a' \right\}.$$

The whole value of V is

$$\frac{4\pi\varrho}{r} \left\{ \begin{array}{l} \psi a - \frac{\varepsilon}{3} a^3 \varphi a - \frac{1}{5}\left(\mu^2 - \frac{1}{3}\right) \varepsilon \frac{a^5}{r^2} \varphi a + \frac{1}{3} \int_0^{a} \mathrm{F} a' \,.\, a'^2 \, d a' \\ \qquad + \frac{1}{5} \frac{\mu^2 - \frac{1}{3}}{r^2} \int_0^{a} \mathrm{F} a' \,.\, a'^4 \, d a' \end{array} \right\}.$$

After writing a for a, and ε_1 for ε, add this to the other value of V, and apply the equations

$$\int_{\mathrm{a}}^{a} \mathrm{F} a' \,.\, a' \, d a' = \int_{\mathrm{a}}^{a} a'^2 \varepsilon' \, d\varphi a' = a^2 \varepsilon \varphi a - \mathrm{a}^2 \varepsilon_1 \varphi \mathrm{a} - \int_{\mathrm{a}}^{a} \varphi a' \, d\left(a'^2 \varepsilon'\right),$$

$$\frac{1}{r} \int_0^{\mathrm{a}} \mathrm{F} a' \,.\, a'^2 \, d a' = \frac{1}{r} \int_0^{\mathrm{a}} a'^3 \varepsilon' \, d\varphi a' = \mathrm{a}^2 \varepsilon_1 \varphi \mathrm{a} - \frac{1}{r} \int_0^{\mathrm{a}} \varphi a' \, d\left(a'^3 \varepsilon'\right),$$

and similar equations for the other integrals; and we shall obtain

$$\mathrm{V} = 4\pi\varrho \left\{ \begin{array}{l} a\varphi_{,}a - \varphi_{,,}a - \mathrm{a}\varphi_{,}\mathrm{a} + \varphi_{,,}\mathrm{a} + \frac{\psi \mathrm{a}}{r} - \frac{1}{3}\left\{ \int_{\mathrm{a}}^{a} \varphi a' \, d(a'^2 \varepsilon') + \frac{1}{r} \int_0^{\mathrm{a}} \varphi a' \, d(a'^3 \varepsilon') \right\} \\ \qquad - \frac{r^2}{5}\left(\mu^2 - \frac{1}{3}\right) \left\{ \int_{\mathrm{a}}^{a} \varphi a' \, d\varepsilon' + \frac{1}{r^5} \int_0^{\mathrm{a}} \varphi a' \, d\left(a'^5 \varepsilon'\right) \right\} \end{array} \right\}.$$

16. To find the equation of equilibrium of a heterogeneous spheroidal mass of fluid, revolving about its axis, with an angular velocity ω.

By the principles of hydrostatics the general equation is $\int \frac{dp}{\varrho'} = \int (\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz)$, ϱ' being the density at the point (x, y, z), p the pressure, and X, Y, Z the sums of the resolved parts of the forces; which are $-\frac{d\mathrm{V}}{dx}, -\frac{d\mathrm{V}}{dy}, -\frac{d\mathrm{V}}{dz}$, and the centrifugal force. Let the axis of z be that of rotation; then the centrifugal force is $\omega^2 x$ along x, and $\omega^2 y$ along y. Let us express ω in terms of the ratio of the centrifugal force at the equator to the equatorial gravity. Call this ratio m, which is small in the case of the earth, being of the same order as ε. Then

$$m = \frac{\omega^2 a^3}{\text{mass}} = \frac{\omega^2 a^3}{4\pi\varrho\psi a}, \text{ or } \omega^2 = \frac{4\pi\varrho m\psi a}{a^3}.$$

Therefore

$$\mathrm{X} = \frac{d\mathrm{V}}{dx} + \frac{4\pi\varrho m\psi a.x}{a^3}, \mathrm{Y} = \frac{d\mathrm{V}}{dy} + \frac{4\pi\varrho m\psi a.y}{a^3}, \mathrm{Z} = \frac{d\mathrm{V}}{dz},$$

and

$$\int \frac{dp}{\varrho'} = \mathrm{V} + \frac{2\pi\varrho m\psi a}{a^3}(1-\mu^2)\,r^2.$$

Now $\int \frac{dp}{\rho'}$ is a constant for a level surface. Hence for any stratum we have

$$\mathrm{C} = \mathrm{V} + \frac{2\pi\rho m\psi a}{a^3}(1-\mu^2)\,r^2.$$

At the surface this is

$$\mathrm{C} = \frac{\psi a}{r} - \frac{1}{3r}\int_0^a \phi\, a'\, d\,(a'^3\varepsilon') - \frac{1}{5}\left(\mu^2 - \frac{1}{3}\right)\frac{1}{r^3}\int_0^a \phi\, a'\, d\,(a'^5\varepsilon') + \frac{m}{2}\frac{\psi a}{a^3}(1-\mu^2)r^2,$$

$$= \frac{1}{3r}\mathrm{M}a - \frac{1}{5r^3}\left(\mu^2 - \frac{1}{3}\right)\mathrm{N}a + \frac{1}{3}\frac{mr^2}{2a^3}\mathrm{M}a\,(1-\mu^2),$$

where

$$\mathrm{M}a = \int_0^a \phi\, a'\, \frac{d\,(a'^3(1-\varepsilon'))}{da'}\, da', \text{ and } \mathrm{N}a = \int_0^a \phi\, a'\, \frac{d\,(a'^5\varepsilon')}{da'}\, da'.$$

For r write $a\,(1-\varepsilon\mu^2)$, then

$$\mathrm{C} = \frac{\mathrm{M}a}{3a}(1+\varepsilon\mu^2) - \frac{1}{5}\left(\mu^2 - \frac{1}{3}\right)\frac{\mathrm{N}a}{a^3} + \frac{1}{6}m\,(1-\mu^2)\frac{\mathrm{M}a}{a}.$$

Equate the coefficients of μ^2, then

$$\frac{\mathrm{M}a}{a}\left(\varepsilon - \frac{m}{2}\right) = \frac{3\,\mathrm{N}a}{5a^3} \ldots . \text{ (B)}.$$

17. By differentiating and changing the sign of $\int \frac{dp}{\rho'}$, we obtain the amount of gravity which acts towards the centre; which, to the order we are now considering, is the same as the whole force of gravity; since the cosine of the angle of the vertical differs from unity only by terms of a higher order.

Consequently

$$g = 4\pi\varrho\left\{\frac{M a}{3 r^2} - \frac{3}{5}\left(\mu^2 - \frac{1}{3}\right)\frac{N a}{r^4} - \frac{m}{3}(1-\mu^2)\frac{r M a}{a^3}\right\},$$

$$= 4\pi\varrho\left\{\frac{M a}{3 a^2}(1 + 2\varepsilon\mu^2) - \frac{3}{5}\left(\mu^2 - \frac{1}{3}\right)\frac{N a}{a^4} - \frac{m}{3}(1-\mu^2)\frac{M a}{a^2}\right\},$$

$$= 4\pi\varrho\frac{M a}{3 a^2}\left\{1 + 2\varepsilon\mu^2 + (1 - 3\mu^2)\left(\varepsilon - \frac{m}{2}\right) - m(1-\mu^2)\right\} \text{(by B.)},$$

$$= 4\pi\varrho\frac{M a}{3 a^2}\left\{1 + \varepsilon - \frac{3m}{2} + \mu^2\left(\frac{5}{2}m - \varepsilon\right)\right\} = G\left\{1 + \sin^2 l\left(\frac{5}{2}m - \varepsilon\right)\right\},$$

where G is the equatorial gravity and l the latitude.

18. From this it appears that up to terms of the 1st order, $R = a(1 - \varepsilon\sin^2 l)$ is the equation of the curve which generates the surface of equilibrium, where the value of ε depends on m, or on the velocity of rotation: but as the coefficients of higher powers of sin l may be considerable, it will be useful to find the surface of equilibrium to a greater degree of exactness. For this purpose we must introduce the fourth power of sin l, whose coefficient will be of the second order. Let the equation of the strata be $a' = r'(1 + \varepsilon'\mu^2 + A'\mu^4)$, or $r' = a'\left(1 - \varepsilon'\mu^2 + (\varepsilon'^2 - A')\mu^4\right)$, where ε' and A' are functions of a' as $\chi\, a'$ and $\theta\, a'$. Then

$$\phi a' = \phi r' + F r'\mu^2 + r^2\frac{d\,(\overline{\chi r}|^2 . \phi' r)}{d r}\frac{\mu^4}{2} + r\,\phi' r\,\theta\, r\,\mu^4 = \phi r + F r.\mu^2 + \Pi r.\mu^4, \text{ suppose.}$$

The $(n+1)$th term of V for an internal point now becomes

$$2\pi\varrho r^n\int_{-1}^{1} d\mu' P_n\left\{\int_r^R \frac{\phi r'}{r'^{n-1}}\, d r' + \mu'^2\int_r^R \frac{F r'}{r'^{n-1}}\, d r' + \mu'^4\int_r^R \frac{\Pi r'}{r'^{n-1}}\, d r'\right\}.$$

By writing for r and R their values in terms of a and a, it is easily found that

$$\int_r^R \frac{\phi r'}{r'^{n-1}}\, d r' \text{ equals } \int_0^a \frac{\phi a'}{a'^{n-1}}\, d a' + \left(\frac{\phi a}{a^{n-2}}\right)\left(-\varepsilon\mu'^2 + \mu'^4(\varepsilon^2 - A') - \mu'^4\frac{n-1}{2}\varepsilon^2\right)$$

$$+ \left(\frac{\phi' a}{a^{n-3}}\right)\varepsilon^2\frac{\mu'^4}{2} - \text{the same functions of a and } \varepsilon_1.$$

And that similar equations are true for the two remaining integrals.

1st. Let $n = 0$, and we have

$$2\pi\varrho\int_{-1}^{1} d\mu'\left\{\begin{array}{l}\int_0^a a'\,\phi a'\, d a' + a^2\phi a\left(-\varepsilon\mu'^2 + \mu'^4\left(\frac{3}{2}\varepsilon^2 - A\right)\right) + a^3\phi' a\frac{\varepsilon^2\mu'^4}{2}\\ + \mu'^2\int_0^a a'.F a'\, d a' + \mu'^2 a^2 F a(-\varepsilon\mu'^2) + \mu'^4\int_0^a a'\,\Pi a'\, d a'\\ - \text{same function of a and } \varepsilon_1,\end{array}\right\},$$

which gives

$$4\pi\varrho\left\{\int_0^a a'\phi a'\, d a' + a^2\phi a\left(-\frac{\varepsilon}{3} + \frac{1}{5}\left(\frac{3}{2}\varepsilon^2 - A\right)\right) + a^3\phi' a\frac{\varepsilon^2}{10} + \frac{1}{3}\int_0^a F a'.a'.d a'\right.$$

$$\left. - a^2 F a\frac{\varepsilon}{5} + \frac{1}{5}\int_0^a a'.\Pi a'\, d a'\right\} - 4\pi\varrho\,(\text{same function of } a_1\ \varepsilon_1 \text{ and } A_1).$$

2nd. Let $n = 2$, and we get

$$\frac{9}{4} 2\pi\varrho\left(\mu^2-\frac{1}{3}\right) r^2 \int_{-1}^{1} d\mu' \left(\mu'^2-\frac{1}{3}\right)\left\{\int_{\mathrm{a}}^{a} \frac{\phi a'}{a'} d a' + (\phi a - \phi \mathrm{a})\left(-\varepsilon' \mu'^2 + \mu'^4\left(\frac{\varepsilon'^2}{2} - \mathrm{A}'\right)\right)\right.$$

$$\left. + (a \phi' a - \mathrm{a} \phi' \mathrm{a}) \frac{\varepsilon'^2 \mu'^4}{2} + \int_{\mathrm{a}}^{a} \frac{\mathrm{F} a'}{a'} d a' + (\mathrm{F} a - \mathrm{F} \mathrm{a})(-\varepsilon' \mu'^2) + \int_{\mathrm{a}}^{a} \Pi a' . a' . d a' \right\},$$

which gives

$$4\pi\varrho\left(\mu^2-\frac{1}{3}\right) r^2 \left\{\phi a \left(-\frac{\varepsilon}{5} + \frac{6}{35}\left(\frac{\varepsilon^2}{2} - \mathrm{A}\right)\right) + a \phi' a \frac{3}{35} \varepsilon^2 + \frac{1}{5}\int_0^a \frac{\mathrm{F} a'}{a'} d a' - \frac{6}{35} \mathrm{F} a . \varepsilon \right.$$

$$\left. + \frac{6}{35}\int_0^a \frac{\Pi a'}{a'} d a' \right\} - 4\pi\varrho\left(\mu^2 - \frac{1}{3}\right) r^2 \text{ (same function of a, } \varepsilon_1 \text{ and } \mathrm{A}_1).$$

3rd. Let $n = 3$, and we have

$$\frac{1225}{64}\left(\mu^4 - \frac{6}{7}\mu^2 + \frac{3}{35}\right) 2\pi\varrho r^4 \int_{-1}^{1} d\mu' \left(\mu'^4 - \frac{6}{7}\mu'^2 + \frac{3}{35}\right)\left\{\int_{\mathrm{a}}^{a} \frac{\phi a'}{a'^3} d a' + \left(\frac{\phi a}{a^2} - \frac{\phi \mathrm{a}}{\mathrm{a}^2}\right)\right.$$

$$\left(-\varepsilon' \mu'^2 - \mu'^4\left(\frac{\varepsilon'^2}{2} + \mathrm{A}'\right)\right) + \left(\frac{\phi' a}{a} - \frac{\phi' \mathrm{a}}{\mathrm{a}}\right) \varepsilon'^2 \frac{\mu'^4}{2} + \int_{\mathrm{a}}^{a} \frac{\mathrm{F} a'}{a'^3} d a' + \left(\frac{\mathrm{F} a}{a^2} - \frac{\mathrm{F} \mathrm{a}}{\mathrm{a}^2}\right)(-\varepsilon' \mu'^2)$$

$$\left. + \int_{\mathrm{a}}^{a} \frac{\Pi a'}{a'^3} d a' \right\},$$

or

$$\left(\mu^4 - \frac{6}{7}\mu^2 + \frac{3}{35}\right) 4\pi\varrho r^4 \left\{-\frac{\phi a}{a^2}\frac{1}{9}\left(\frac{\varepsilon^2}{2} + \mathrm{A}\right) + \frac{\phi' a}{a}\frac{\varepsilon^2}{18} - \frac{\varepsilon}{9}\frac{\mathrm{F} a}{a^2} + \frac{1}{9}\int_0^a \frac{\Pi a'}{a'^3} d a' \right\}$$

$$- \left(\mu^4 - \frac{6}{7}\mu^2 + \frac{3}{35}\right) 4\pi\varrho r^4 \text{ (same function of a, } \varepsilon_1 \text{ and } \mathrm{A}_1).$$

19. To the sum of these must be added V for the inner spheroid, for which we shall have to find the value of V generally, when the attracted point is without the body.

The general term of the series for V is

$$\frac{2\pi\varrho}{r^{n-1}}\int_{-1}^{1} d\mu' \mathrm{P}_{n-2}\left\{\int_0^{\mathrm{R}} r'^n . \phi r' . d r' + \mu'^2 \int_0^{\mathrm{R}} r'^n \mathrm{F} r' . d r' + \mu'^4 \int_0^{\mathrm{R}} r'^n . \Pi r' . d r' \right\},$$

and

$$\int_0^{\mathrm{R}} r'^n \phi r' d r' = \int_0^a a'^n . \phi a' . d a' + a^{n+1} \phi a \left(-\varepsilon \mu^2 + \mu^4 (\varepsilon^2 - \mathrm{A}) + \mu^4 \frac{n}{2} \varepsilon^2\right)$$

$$+ a^{n+2} \phi' a \frac{\varepsilon^2 \mu^4}{2};$$

and similar equations hold for the other functions.

1st. Let $n = 2$, then we have

$$\frac{2\pi\varrho}{r}\int_{-1}^{1} d\mu' \left\{\int_0^a a'^2 \phi a' d a' + a^3 \phi a \left(-\varepsilon \mu'^2 + \mu'^4 (2\varepsilon^2 - \mathrm{A})\right) + a^4 \phi' a \frac{\varepsilon^2}{10}\right.$$

$$\left. + \int_0^a a'^2 \mathrm{F} a' d a' + a^3 \mathrm{F} a (-\varepsilon \mu'^2) + \int_0^a a'^2 \Pi a' d a' \right\},$$

which is equal to

$$\frac{4\pi\varrho}{r}\left\{\int_0^a a'^2 \phi a' d a' + a^3 \phi a \left(-\frac{\varepsilon}{3} + \frac{1}{5}(2\varepsilon^2 - \mathrm{A})\right) + a^4 \phi' a \frac{\varepsilon^2}{10} + \frac{1}{3}\int_0^a a'^2 \mathrm{F} a' . d a' \right.$$

$$\left. - \frac{1}{5} a^3 \mathrm{F} a . \varepsilon + \frac{1}{5}\int_0^a a'^2 \Pi a' d a' \right\}.$$

2nd. Let $n = 4$, and the term is

$$\frac{9}{4}\left(\mu^2-\frac{1}{3}\right)\frac{2\pi\varrho}{r^3}\int_{-1}^{1}d\mu'\left(\mu'^2-\frac{1}{3}\right)\left\{\int_0^a a'^4\varphi a'\,d a'+a^5\varphi a\left(-\varepsilon\mu'^2+\mu'^4(3\varepsilon^2-\mathrm{A})\right)\right.$$

$$\left.+a^6\varphi' a\frac{\varepsilon^2\mu'^4}{2}+\int_0^a a'^4\,\mathrm{F}\,a'\,d a'-a^5\,\mathrm{F}\,a\,\varepsilon\mu'^2+\int_0^a a'^4\,\Pi\,a'\,d a'\right\},$$

or

$$4\pi\varrho\frac{\mu^2-\frac{1}{3}}{r^3}\left\{a^5\varphi a\left(-\frac{\varepsilon}{5}+\frac{6}{35}(3\varepsilon^2-\mathrm{A})\right)+a^6\varphi' a\frac{3}{35}\varepsilon^2+\frac{1}{5}\int_0^a a'^4\,\mathrm{F}\,a'\,d a'\right.$$

$$\left.-\frac{6}{35}a^5\,\mathrm{F}\,a\,.\,\varepsilon+\frac{6}{35}\int_0^a a'^4\,\Pi\,a'\,d a'\right\}.$$

3rd. Let $n = 6$, and we get

$$\frac{1225}{64}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right)\frac{4\pi\rho}{r^5}\int_{-1}^{1}d\mu'\left(\mu'^4-\frac{6}{7}\mu'^2+\frac{3}{35}\right)\left\{\int_0^a a'^6\varphi\,a'\,d a'\right.$$

$$+a^7\varphi a\left(-\varepsilon\mu'^2+\mu'^4(4\varepsilon^2-\mathrm{A})\right)+a^8\varphi' a\frac{\varepsilon^2\mu'^4}{2}+\int_0^a a'^6\,\mathrm{F}\,a'\,d a'-a^7\,\mathrm{F}\,a\,\varepsilon\mu'^2$$

$$\left.+\int_0^a a'^6\,\Pi\,a'\,d a'\right\},$$

which is

$$\frac{4\pi\varrho}{r^5}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right)\left\{a^7\varphi a\frac{1}{9}(4\varepsilon^2-\mathrm{A})+a^8\varphi' a\frac{1}{18}\varepsilon^2-\frac{\varepsilon}{9}a^7\,\mathrm{F}\,a+\frac{1}{9}\int_0^a a'^6\,\Pi\,a'\,d a'\right\}.$$

20. In these three terms write a for a, ε_1 for ε, and A_1 for A, and add them to the other value of V, and apply the equations

$$\int_0^a a'^2\,\Pi\,a'\,d a'=\frac{1}{2}\int_0^a a'^4\,d\left(\chi\,a'^2\,\varphi'\,a'\right)+\int_0^a a'^3\,\theta\,a'\,d\left(\varphi\,a'\right)=\frac{1}{2}a^4\varepsilon^2\varphi' a+a^3\mathrm{A}\,\varphi\,a$$

$$-2\int_0^a a'^3\chi\,a'^2\,.\,d\,\varphi\,a'-\int_0^a\varphi\,a'\,d\left(a'^3\mathrm{A}'\right)=\frac{1}{2}a^4\varepsilon^2\varphi' a+a^3\mathrm{A}\,\varphi\,a-2\,a^3\varepsilon^2\varphi\,a$$

$$+2\int_0^a\varphi\,a'\,d\left(a'^3\varepsilon'^2\right)-\int_0^a\varphi\,a'\,d\left(a'^3\mathrm{A}'\right),$$

and similar equations for the other integrals, and the value of V for an internal point becomes

$$4\pi\varrho\left\{\begin{array}{l}\displaystyle\int_{\mathrm{a}}^a\varphi a'\frac{d\left(a'^2\left\{\frac{1}{2}-\frac{1}{3}\varepsilon'+\frac{1}{5}\left(\frac{3}{2}\varepsilon'^2-\mathrm{A}'\right)\right\}\right)}{d a'}d a'+\frac{1}{3r}\int_0^{\mathrm{a}}\varphi\,a'\frac{d\left(a'^3\left\{1-\varepsilon'+\frac{3}{5}(2\varepsilon'^2-\mathrm{A}')\right\}\right)}{d a'}d a'\\ \displaystyle-\frac{1}{5}\left(\mu^2-\frac{1}{3}\right)\left(r^2\int_{\mathrm{a}}^a\varphi\,a'\frac{d\left(\varepsilon'-\frac{3}{7}(\varepsilon'^2-2\mathrm{A}')\right)}{d a'}d a'+\frac{1}{r^3}\int_0^{\mathrm{a}}\varphi a'\frac{d\left(a'^5\left\{\varepsilon'-\frac{6}{7}(3\varepsilon'^2-\mathrm{A}')\right\}\right)}{d a'}d a'\right)\\ \displaystyle+\frac{1}{9}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right)\left(r^4\int_a^{\mathrm{a}}\varphi a'\frac{d\frac{1}{a'^2}\left(-\frac{\varepsilon'^2}{2}-\mathrm{A}'\right)}{d a'}d a'+\frac{1}{r^5}\int_0^{\mathrm{a}}\varphi a'\frac{d\left(a'^7(4\varepsilon'^2-\mathrm{A}')\right)}{d a'}d a'\right)\end{array}\right\}$$

$$=4\pi\varrho\left\{\mathrm{M}'a-\mathrm{M}'\mathrm{a}+\frac{1}{3r}\mathrm{M}\,\mathrm{a}+\frac{1}{5}\left(\mu^2-\frac{1}{3}\right)\left(r^2\mathrm{N}'a-r^2\mathrm{N}'\mathrm{a}+\frac{1}{r^3}\mathrm{N}\,\mathrm{a}\right)\right.$$

$$\left.+\frac{1}{9}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right)\left(r^4\mathrm{P}'a-r^4\mathrm{P}'\mathrm{a}+\frac{1}{r^5}\mathrm{P}\,\mathrm{a}\right)\right\},$$

where the functions M, N, P, &c. are substituted for the corresponding integrals.

21. The equation of equilibrium will be formed exactly as in (16.), except that the expression for the velocity must be found more accurately. By (17.), the force of gravity at the equator is $4\pi\varrho\frac{\mathrm{M}a}{3a^2}\left(1+\varepsilon-\frac{3m}{2}\right)$; and the centrifugal force is $\omega^2 a$; therefore $m=\dfrac{\omega^2 a}{4\pi\rho\frac{\mathrm{M}a}{3a^2}\left(1+\varepsilon-\frac{3m}{2}\right)}$, and $\omega^2=4\pi\varrho\frac{\mathrm{M}a}{3a^3}\left(m+m\varepsilon-\frac{3m^2}{2}\right)$.

The equation of equilibrium becomes

$$\int\frac{dp}{\rho'}=\mathrm{C}=\mathrm{V}+2\pi\varrho r^2(1-\mu^2)\frac{\mathrm{M}a}{3a^3}m\left(1+\varepsilon-\frac{3m}{2}\right),$$

which at the surface becomes

$$\mathrm{C}'=\frac{\mathrm{M}a}{3r}-\frac{1}{5}\left(\mu^2-\frac{1}{3}\right)\frac{\mathrm{N}a}{r^3}+\frac{1}{9}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right)\frac{\mathrm{P}a}{r^5}+\frac{1}{2}(1-\mu^2)r^2\frac{\mathrm{M}a}{3a^3}\left(m+m\varepsilon-\frac{3m^2}{2}\right),$$

$$=\frac{\mathrm{M}a}{3a}(1+\varepsilon\mu^2+\mathrm{A}\mu^4)-\frac{1}{5}\left(\mu^2-\frac{1}{3}\right)(1+3\varepsilon\mu^2)\frac{\mathrm{N}a}{a^3}+\frac{1}{9}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right)\frac{\mathrm{P}a}{a^5}$$

$$+\frac{1-\mu^2}{2}(1-2\varepsilon\mu^2)\frac{\mathrm{M}a}{3a^3}\left(m+m\varepsilon-\frac{3m^2}{2}\right).$$

Equate the coefficients of μ^2 and those of μ^4, and we have two equations for determining ε and A, thus showing that equilibrium is possible. These are,

$$\frac{\mathrm{M}a}{3a}\left(\varepsilon-\frac{m}{2}-\frac{3}{2}m\varepsilon+\frac{3}{4}\mu^2\right)-\frac{\mathrm{N}a}{a^3}\left(\frac{1}{5}-\frac{\varepsilon}{5}\right)-\frac{2}{21}\frac{\mathrm{P}a}{a^5}=0,$$

and

$$\frac{\mathrm{M}a}{3a}(\mathrm{A}+m\varepsilon)-\frac{\mathrm{N}a}{a^3}\frac{3\varepsilon}{5}+\frac{1}{9}\frac{\mathrm{P}a}{a^5}=0;$$

whence

$$\frac{\mathrm{M}a}{3a}\left(\varepsilon-\frac{m}{2}+\frac{6\mathrm{A}}{7}+\frac{3}{4}m^2+\frac{m\varepsilon}{7}-\frac{11}{7}\varepsilon^2\right)=\frac{\mathrm{N}a}{5a^3},$$

and

$$\frac{\mathrm{M}a}{3a}\left(\mathrm{A}-3\varepsilon^2+\frac{5m\varepsilon}{2}\right)=-\frac{1}{9}\frac{\mathrm{P}a}{a^5}.$$

22. The resultant attraction in the direction of r, is obtained by differentiating $\int\frac{dp}{\rho'}$, as found in (21.), and changing its sign. This produces

$$4\pi\varrho\left\{\frac{\mathrm{M}a}{3r^2}-\frac{3}{5}\left(\mu^2-\frac{1}{3}\right)\frac{\mathrm{N}a}{r^4}+\frac{5}{9}\left(\mu^4-\frac{6}{7}\mu^2+\frac{3}{35}\right)\frac{\mathrm{P}a}{r^6}-r(1-\mu^2)\left(m+m\varepsilon-\frac{3m^2}{2}\right)\frac{\mathrm{M}a}{3a^3}\right\},$$

since the terms arising from differentiating the expression for V in (20.) with respect to a, cancel each other.

For r write its value, and arrange the result according to powers of μ, and this formula becomes

$$4\pi\varrho\left\{\begin{array}{l}\frac{\mathrm{M}a}{3a^2}\left(1-m+m\varepsilon-\frac{3m^2}{2}+\mu^2\left(2\varepsilon+m+2m\varepsilon-\frac{3m^2}{2}\right)+\mu^4(2\mathrm{A}+\varepsilon^2-m\varepsilon)\right)\\ \quad-\frac{\mathrm{N}a}{a^4}\left(-\frac{1}{5}+\mu^2\left(\frac{3}{5}-\frac{4}{5}\varepsilon\right)+\mu^4\frac{12}{5}\varepsilon\right)+\frac{\mathrm{P}a}{a^6}\left(\frac{1}{21}-\mu^2\frac{10}{21}+\mu^4\frac{5}{9}\right)\end{array}\right\}$$

$$=4\pi\varrho\frac{\mathrm{M}a}{3a^2}\left\{1+\varepsilon-\frac{3m}{2}-\frac{27}{14}m\varepsilon+\frac{9}{4}m^2-\frac{2}{7}\varepsilon^2+\frac{3}{7}\mathrm{A}\right.$$

$$\left.+\mu^2\left(\frac{5}{2}m-\varepsilon+\frac{72}{7}m\varepsilon-\frac{15}{4}m^2-\frac{29}{7}\varepsilon^2+\frac{12}{7}\mathrm{A}\right)+\mu^4\left(4\varepsilon^2-3\mathrm{A}-\frac{15}{2}m\varepsilon\right)\right\},$$

by applying the two equations deduced in last article. The total attraction normal to the surface, or the force of gravity, is found from this by dividing by the cosine of the angle of the vertical, or by $1 - 2\,\varepsilon^2 (\mu^2 - \mu^4)$; whence

$$g = 4\,\pi\,\varrho\,\frac{M\,a}{3\,a^2}\left\{1 + \varepsilon - \frac{3\,m}{2} - \frac{27}{14}\,m\,\varepsilon + \frac{9}{4}\,m^2 - \frac{2}{7}\,\varepsilon^2 + \frac{3}{7}\,A\right.$$
$$\left. + \mu^2\left(\frac{5}{2}\,m - \varepsilon + \frac{72}{7}\,m\,\varepsilon - \frac{15}{4}\,m^2 - \frac{15}{7}\,\varepsilon^2 + \frac{12}{7}\,A\right) + \mu^4\left(2\,\varepsilon^2 - 3\,A - \frac{15}{2}\,m\,\varepsilon\right)\right\}.$$

Let λ be the sine of the real latitude; then as $\mu^2 = \lambda^2 - 4\,\varepsilon\,\overline{\lambda^2 - \lambda^4}$, we get

$$g = 4\,\pi\,\varrho\,\frac{M\,a}{3\,a^2}\left\{1 + \varepsilon - \frac{3\,m}{2} - \frac{27}{14}\,m\,\varepsilon + \frac{9}{4}\,m^2 - \frac{2}{7}\,\varepsilon^2 + \frac{3}{7}\,A\right.$$
$$\left. + \lambda^2\left(\frac{5}{2}\,m - \varepsilon + \frac{2}{7}\,m\,\varepsilon - \frac{15}{4}\,m^2 + \frac{13}{7}\,\varepsilon^2 + \frac{12}{7}\,A\right) - \lambda^4\left(2\,\varepsilon^2 + 3\,A - \frac{5}{2}\,m\,\varepsilon\right)\right\}.$$

Let G represent the equatorial gravity; then

$$g = G\left\{1 + \lambda^2\left(\frac{5}{2}\,m - \varepsilon - \frac{26}{7}\,m\,\varepsilon + \frac{20}{7}\,\varepsilon^2 + \frac{12}{7}\,A\right) - \lambda^4\left(2\,\varepsilon^2 + 3\,A - \frac{5}{2}\,m\,\varepsilon\right)\right\}$$
$$= G\left\{1 + \sin^2 l\left(\frac{5}{2}m - \varepsilon - \frac{17}{14}\,m\,\varepsilon + \frac{6}{7}\,\varepsilon^2 - \frac{9}{7}\,A\right) + \sin^2 l \cos^2 l\left(2\,\varepsilon^2 + 3\,A - \frac{5}{2}m\,\varepsilon\right)\right\},$$

which is an extension of CLAIRAUT's theorem.

23. In this process A indicates the amount of deviation of the required surface from the surface represented by $r = a\,(1 - \varepsilon\,\mu^2)$. If the equation had been taken $a = r\left\{1 + e\,\mu^2 - \left(\frac{3}{2}\,e^2 + B\right)\left(\mu^4 - \mu^2\right) + e^2\,\mu^4\right\}$, B would have been the index of deviation from an elliptic spheroid.

To apply CLAIRAUT's theorem to this surface, we have

$$\varepsilon = e + \frac{3}{2}\,e^2 + B, \text{ and } A = -\frac{e^2}{2} - B;$$

whence

$$g = G\left\{1 + \sin^2 l\left(\frac{5}{2}\,m - e + \frac{2}{7}\,B - \frac{17}{14}\,m\,e\right) - \sin^2 l \cos^2 l\left(\frac{5}{2}\,m\,e - \frac{e^2}{2} + 3\,A\right)\right\},$$

which is the same expression as that obtained by Mr. AIRY in the Philosophical Transactions, 1826, except that instead of e or $\frac{a-c}{a}$, the symbol is used to represent $\frac{a-c}{c}$.

24. The circumstance of the terms arising from the differentiation of V with respect to a vanishing, affords an easy method of extending CLAIRAUT's theorem indefinitely, without calculating the value of V.

It may be shown independently, that these terms cancel each other in all cases.

The $(n+1)$th term of V for the portion including the point is

$$2\,\pi\,\varrho\,r^n \int_{-1}^{1} d\,\mu'\,P_n\left\{\int_r^R \frac{\phi\,r'}{r'^{n-1}}\,d\,r' + \mu'^2 \int_r^R \frac{F\,r'}{r'^{n-1}}\,d\,r' + \mu'^4 \int_r^R \frac{\Pi\,r'}{r'^{n-1}}\,d\,r' + \dots\right\}.$$

The corresponding term for the other portion is

$$\frac{2\,\pi\,\rho}{r^{n+1}} \int_{-1}^{1} d\,\mu'\,P_n\left\{\int_0^r r'^{n+2}\,\phi r'\,d\,r' + \mu'^2 \int_0^r r'^{n+2}\,F\,r'\,d\,r' + \mu'^4 \int_0^r \Pi\,r'.r'^{n+2}\,d\,r' + \dots\right\};$$

and it is evident, by inspection of these functions, that that portion of the sum of their partial differential coefficients, which arises from differentiating with respect to r under the signs of integration, is equal to zero. This being the case, it is not necessary to know the forms of the functions $\mathrm{M}\,a$, $\mathrm{N}\,a$, $\mathrm{P}\,a$, &c., nor their numerical coefficients; but only the functions of μ, by which they are respectively multiplied; and these are LAPLACE's coefficients.

Thus the equation of equilibrium at the surface would be

$$\mathrm{C} = \frac{\mathrm{M}\,a}{3\,r} - \frac{1}{5}\left(\mu^2 - \frac{1}{3}\right)\frac{\mathrm{N}\,a}{r^3} + \frac{1}{9}\left(\mu^4 - \frac{6}{7}\mu^2 + \frac{3}{35}\right)\frac{\mathrm{P}\,a}{r^5} - \left(\mu^6 - \frac{15}{11}\mu^4 + \frac{5}{11}\mu^2 - \frac{5}{231}\right)\frac{\mathrm{Q}\,a}{r^7}$$

$$+ \frac{1}{2}(1 - \mu^2)\,r^2\,m\,\mathrm{G}, \text{ where } \frac{1}{r} = \frac{1}{a}(1 + \varepsilon\,\mu^2 + \mathrm{A}\,\mu^4 + \mathrm{D}\,\mu^6), \text{ suppose.}$$

Expand r, recollecting that $\mathrm{N}\,a$, $\mathrm{P}\,a$, and $\mathrm{Q}\,a$ are of the 1st, 2nd, and 3rd orders respectively, and we have three equations to determine ε, A and D.

By differentiating C with respect to r, and eliminating $\mathrm{N}\,a$, $\mathrm{P}\,a$, and $\mathrm{Q}\,a$ by these three equations, we have the resolved force in r, which divided by the cosine of the angle of the vertical gives g exactly as in (23.).

It is evident that this may be carried on indefinitely; and to any order, without finding g for the next lower order.